pressures and temperatures during the central portion of the cycle but decreases them during the last 35% of the cycle; increasing the effusion parameter K decreases the pressures and temperatures during all parts of the cycle.

The beam density at the detector was calculated numerically for both Eqs. (7) and (11). Typical predicted values are compared with measured values in Fig. 1. The results justify the use of the simplified model for cases in which the most probable time of flight is less than about 5% of the cycle period.

B. Experimental results

Data taken to demonstrate the increasing skimmer-interference effects due to increasing mass-flow rates are presented in Figs. 2a–2c. It is seen that the beam density decreases dramatically, particularly in the vicinity of the top dead center, as the mass-flow rate increases. An analysis of source-chamber flows indicates that the observed lag of the source-chamber pressure peak behind the source pressure peak is determined by the ratio of the booster-pump characteristic time and the source-cycle period. The smaller source-chamber pressure peak appearing in the early part of the cycle is due to a superpositioning of residual gas from the previous cycle and gas from the present cycle.

Consequences of varying the nozzle-skimmer distance x are shown in Fig. 3. For small nozzle-skimmer distances, strong skimmer interference occurs near the peak density of the cycle; for large nozzle-skimmer distances, the beam density is attenuated throughout the cycle.

Beam densities observed for three different engine speeds are compared in Fig. 4. It is seen that the peak beam density increases with engine speed, which speed is inversely proportional to the effusion parameter K.

Discussions

Agreement between the measured and predicted signals is good except near the end of the cycle (see Fig. 1). The relatively low beam density observed near the end of the cycle might be due to background scattering resulting from the relatively high source-chamber background density during this part of the cycle.

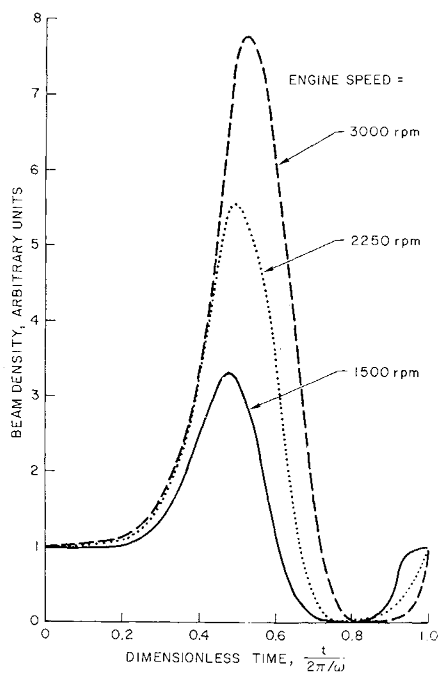

Fig. 4 Measured values of beam density for several engine speeds. The signals are measured from their minima and normalized using the values evaluated at $t = 0$.

The measured peak source pressures (Fig. 2) are substantially smaller than the predicted values. It is possible that the measured peak pressure was decreased by a) gas leakage between the piston and the cylinder wall and b) heat transfer from the gas to the cylinder wall.

Continuations of these studies have shown that skimmer interference affects the relative densities in a multicomponent beam less than it affects the absolute density. For details of these additional studies, see Refs. 4–6.

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Vortices Induced in a Jet by a Subsonic Cross Flow

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Nomenclature

d = nozzle diameter
$\mathbf{I}$ = impulse
K' = empirical constant
K = vortex strength constant
U = velocity
x,y,z = coordinates in Fig. 1
$2y_0$ = vortex spacing
Y_v = vortex spacing constant
μ = angle between jet trajectory and freestream direction
ξ,η,ζ = coordinates in Fig. 1
ρ = density
σ = freestream-to-jet exit velocity ratio
ϕ = velocity potential

Subscripts

∞ = freestream conditions
e = jet exit conditions

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1. Introduction

SIGNIFICANT interference effects sometimes arise between a jet exhausting transversely to the freestream and aerodynamic control surfaces placed aft of the jet nozzle. Missiles equipped with roll control jets placed near the nose encounter effects of this sort. Interference between a lift jet and the empennage of VTOL aircraft during transition flight also falls in this category. In order to estimate the magnitude of such interference effects, a knowledge of jet properties as a function of distance along the jet trajectory is required. For this purpose, a simple, semiempirical model of a jet plume in a cross flow has been developed, valid in a region where the jet axis is almost aligned with the freestream. The minimum distance downstream of the nozzle at which the model is expected to be valid is defined in terms of similarity variables. Two empirical constants are introduced in the course of the analysis. Their numerical values are calculated from data reported in Ref. 1.

2. Proposed Model

It is known that a jet in a subsonic cross flow contains two counter-rotating vortices.[1-3] Pratte and Baines[1] indicate that at large distances downstream of the nozzle, the axial velocity in the jet is almost equal to the freestream velocity, and that the jet plume is almost aligned with the freestream. They refer to this region as the "vortex zone," because the counter-rotating vortices still persist, although their strength decays due to viscous dissipation. The model developed applies to this region. Since jet and freestream directions differ but little, perturbation velocities are assumed to be small compared to the freestream velocity. The flow is then analyzed as an unsteady flow in the y-z plane (Fig. 1), with the vortices being convected downstream at the freestream velocity, U_∞. In the y-z plane, the plume is represented by two-counter-rotating vortices located at $(-y_o,z_o)$ and (y_o,z_o), and connected by a line of discontinuity in the potential, as illustrated in Fig. 2.

The impulse necessary to generate the vortex motion instantaneously from rest is given by[4]

$$\mathbf{I} = -\oint \rho\phi\mathbf{k}ds \tag{1}$$

where $\mathbf{k}$ is a unit vector in the z direction.

The integral in Eq. (1) is to be taken around the path C enclosing both vortices and the connecting line. The velocity potential for the vortices may be written as

$$\phi = \Gamma/2\pi\{\tan^{-1}[(z - z_o)/(y + y_o)] - \tan^{-1}[(z - z_o)/(y - y_o)]\} \tag{2}$$

Substituting Eq. (2) into Eq. (1) and performing the integration,

$$\mathbf{I} = \mathbf{k}(2\rho\Gamma y_o) \tag{3}$$

The rate of change of $\mathbf{I}$ with respect to time is equal to the net force which must be applied to the vortices and connecting sheet system in order to generate the fluid motion instantaneously from rest.[4] Thus,

$$\mathbf{F} = \mathbf{k}(2\rho)(d/dt)(\Gamma y_o) \tag{4}$$

It will now be postulated that vortex strength and separation must vary in such a way that the net force on the vortices and connecting line is zero. This assumption is often made in the analysis of separated flow about a slender body at high incidence.[5,6] Using Eq. (4), this requirement implies

$$\Gamma = K'/y_o \tag{5}$$

where K' is an integration constant. A similar model has been used by Turner[7] to describe the vortex pair in a buoyant plume in a cross flow. In Turner's model, however, it is assumed that the vortex strength remains constant, and that the force acting on the vortex system is the buoyancy force. In the present model, Eq. (5) indicates that if the vortices draw apart, their strength must decrease. Since no viscous dissipation has been included, it is difficult to explain what happens to the vorticity "released" by a decrease in the vortex strength. In Bryson's[6] model of separated flow about a body of revolution at high incidence, the increase in vortex strength is explained by a feeding process, wherein vorticity generated in the boundary layer is carried via the connecting sheet to the vortex core. No such mechanism may be postulated in the current instance where no solid boundaries are present. A possible explanation might be to say that as the vortex strength on one side decreases, a small amount of vorticity is carried via the connecting sheet to the plane of symmetry, to be cancelled there by an equal amount of vorticity of opposite sign arriving from the vortex on the other side.

The vortices shown in Fig. 2 convect upward at the velocity induced by one vortex at the other's location. Thus,

$$dz_o/dt = \Gamma/4\pi y_o$$

and, since the vortices were assumed to convect downstream at U_∞,

$$\tan\mu = dz_o/dx = \Gamma/4\pi U_\infty y_o$$

where the angle μ is defined in Fig. 1. Since the angle μ is small, the aforementioned may be written as

$$dz_o/d\xi = \Gamma/4\pi U_\infty y_o \tag{6}$$

If the dependence of the vortex spacing y_o on the coordinate ξ is given, Eqs. (5) and (6) predict the change in Γ with ξ and the jet trajectory. However, it is advantageous to first write the equations in terms of similarity variables.

Pratte and Baines[1] find that they can correlate their data for jet trajectory and thickness using variables scaled by the jet-to-freestream velocity ratio. Based on this fact, the following similarity variables are defined:

$$\chi = \sigma_e\xi/d_e,\ X = \sigma_e x/d_e,\ Y = \sigma_e y/d_e,\ Z = \sigma_e z/d_e \tag{7}$$

where

$$\sigma_e = U_\infty/U_e$$

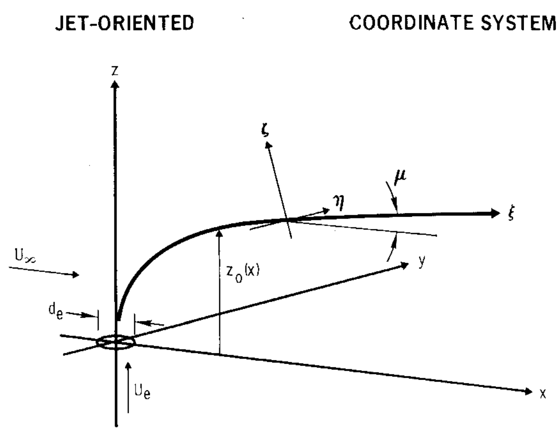

Fig. 1 Jet oriented coordinate system.

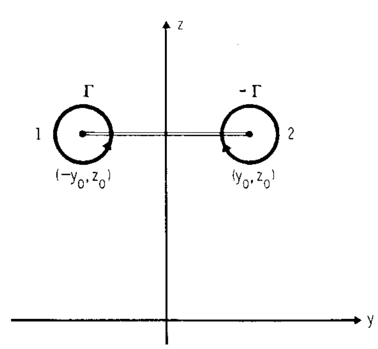

Fig. 2 Vortices in the crossflow plane.

Pratte and Baines find that the jet cross section grows as $\chi^{1/3}$, as for a jet in a coaxial stream (Figs. 7 and 9 in Ref. 1). It therefore seems reasonable to assume that the vortex spacing also grows as $\chi^{1/3}$, so that

$$Y_o = Y_v\chi^{1/3} \tag{8}$$

where Y_v is a universal constant. Substituting Eq. (5) into Eq. (6) and writing the result in similarity variables yields

$$dZ_o/d\chi = (K'\sigma_e^2/4\pi U_\infty d_e^2)(1/Y_v^2\chi^{2/3}) \tag{9}$$

The data of Pratte and Baines show that Z_o is a universal function of χ, and consequently Eq. (9) requires that

$$K'\sigma_e^2/4\pi U_\infty d_e^2 = K \tag{10}$$

where K is another universal constant. Furthermore, integrating Eq. (9) leads to the result

$$Z_o = (3K/Y_v^2)\chi^{1/3} \tag{11}$$

where the constant of integration has been arbitrarily set equal to zero. The jet trajectory predicted by this equation is verified by the data of Pratte and Baines, who measured

$$Z_o = (\text{const})\chi^{1/3}$$

Finally, it is possible to write Eq. (5) in terms of similarity variables. Using Eq. (10), the result is

$$\Gamma^* = K/Y_v\chi^{1/3} \tag{12}$$

where

$$\Gamma^* = (\Gamma/4\pi U_\infty d_e)\sigma_e \tag{13}$$

Equations (12) and (13) show that the product of normalized vortex strength $(\Gamma/4\pi U_\infty d_e)$ and σ_e is a universal function of the similarity variable $\chi^{1/3}$.

In summary, starting from a vortex spacing based on data, the model predicts the correct jet trajectory to within an empirical constant. Indirectly, at least, this appears to verify the proposed relation between vortex strength and spacing. The results derived are expected to hold within the vortex zone, which has been found to lie[1] downstream of the value $\chi = 5$. Two empirical constants have been introduced so far, K and Y_v, and their numerical values must be found from data. From Fig. 9 of Ref. 1,

$$Y_v = 1.45$$

From Eq. (11) and Fig. 5 of Ref. 1,

$$3K/Y_v^2 = 1.63$$

Equation (12) then yields the following expression for the correlated vortex strength:

$$\Gamma^* = (0.79)/\chi^{1/3} \tag{14}$$

3. Conclusions

A model to represent the vortex zone within a jet in a subsonic cross flow has been developed. This vortex zone lies downstream[1] of the correlated jet axial distance $\chi = 5$. The analysis yields an expression for the strength of the counterrotating vortices in the jet as a function of distance along the jet trajectory. A correlated form of this vortex strength also results from the analysis. The model is self-consistent, since matching the measured jet spread leads to the correct form for the jet trajectory. Direct comparison of predicted and measured vortex strengths, however, cannot be made at present, due to a lack of data.

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Heat Transfer with Nonlinear Boundary Conditions via a Variational Principle

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I. Introduction

In this study the authors develop a technique based on a variational principle that possesses a Hamiltonian structure, and which is different from Biot's Principle.[1] The variational principle developed here corresponds to the general problem of heat conduction with finite wave speeds. The transition to the classical situation, viz., where the speed of propagation of the thermal disturbance is infinite, is accomplished by allowing the relaxation time to approach zero.

Biot,[1] by applying variational techniques, developed a method of formulating heat-transfer problems involving nonlinear boundary conditions. Biot combined the concepts of thermal potential, dissipation function, and generalized thermal forces with the variational principle to develop a powerful technique for solving nonlinear boundary value problems.

It should be pointed out that the variational principle developed here is neither a quasi-variational formulation of the problem, as is Biot's, nor a restricted variational principle, as that of Glansdorff and Prigogine.[2] The motivation here is to develop a variational technique applicable to nonlinear heat conduction problems within the framework of the classical calculus of variations. This is accomplished by considering the problem of heat conduction with finite wave speeds, Eq. (2), and making use of the exact Lagrangian.

The main drawback to this method is that, at this time, the error of the approximate solution cannot be controlled. It should, however, be noted that, to the authors' knowledge, no method of approximate solution in the physics of irreversible phenomena exists in which the error of the approximate solution can be controlled. Examples of other methods are Biot's, Galerkin's, and the integral method.

The concepts of generalized coordinates and the method of partial integration are the basic tools of this theory. By

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